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ABSTRACT

Several studies have shown the difficulties students encounter in making sense of situations involving rate of change. This study concerns how students discover errors and refine their knowledge when working with rate of change. The part of the study reported here concerns the responses of four precalculus students to a task which asked them to sketch a distance-time graph showing slowing down then speeding up. These four students drew the same incorrect graph. This report is about how they discovered and corrected the error. Two general conclusions from this study are that students use a variety of resources to address rate of change, and that slope and changes over intervals are both powerful ways for precalculus students to think about rate of change. An instructional implication of this study is that calculus and precalculus teachers should provide opportunities for students to use their knowledge of slope and changes over intervals to construct knowledge of rate of change. Teachers should notice the knowledge students use to make sense of situations and help students use that knowledge to construct new mathematical knowledge. Contains 41 references. Appendices contain statements of tasks and graphs. (Author/PVD)

Growth of knowledge of rate in four precalculus students

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Abstract

Several studies show difficulties students encounter in making sense of situations involving rate of change. There is less research on how students discover errors and refine their knowledge when working with rate of change. The study reported here concerns the latter.

This study is part of a larger study about knowledge of rate of change of precalculus, calculus, and postcalculus students. The part of the study reported here concerns the responses of four precalculus students to a task which asked students to sketch a distance-time graph showing "slowing down" and then "speeding up". These four students drew the same incorrect graph. This report is about how they discovered and corrected their error.

The four precalculus students drew a graph that showed "steady pace". All of them discovered their error - one before I asked her any questions; one when I asked him to explain how his graph showed slowing down and speeding up; one after she explained how her graph showed slowing down and speeding up, created a table of values for her table and tried to explain how it showed slowing down and speeding up; and one when she began working on a different task which called for a graph that showed "steady pace".

The first student came to see that slope and shape of the distance-time graph give information about speed and used this information to then draw a correct graph. The second student carefully examined and explained his graph to see that it represented steady pace. He then drew a correct graph and gave an explanation of how it showed slowing down and speeding up. For both graphs he focused on change in the height of the graph over equal intervals of time to make decisions about the rate of change associated with each graph.

The third student justified her initial graph by saying "If they're steadily slowing down and they're steadily speeding up, it should just be straight lines.". When she made a table of values for her graph and tried to explain how it showed slowing down and speeding up, she saw that the distance was changing the same number of feet each second and concluded that it showed "steady pace". She then created a correct graph by carefully constructing appropriate changes in distance for each second to get a graph that showed slowing down and then speeding up.

The fourth student did not realize her error until she got to the second task, which called for a graph that showed "steady pace". She noticed then that her graph for the first task showed the same distance being traveled each second and proceeded to create a correct graph by thinking in terms of the time needed to cover a certain distance to show slowing down and then speeding up.

Of the four students, one used slope of graph, one used reference to changes on another graph, and two referred to relationships between distance and time to discover and correct their error. Two general conclusions from this study are that students use a variety of resources to address rate of change and that slope and changes over intervals are both powerful ways for precalculus students to think about rate of change.

An instructional implication of this study it that calculus and precalculus teachers should provide opportunities for students to use their knowledge of slope and changes over intervals to construct knowledge of rate of change. More generally, teachers should notice knowledge students use to make sense of situations and help students use that knowledge to construct new mathematical knowledge.

Introduction

Many studies show that students struggle in their attempts to learn calculus. Some show calculus students having trouble forming and manipulating algebraic expressions for relationships between changing quantities (Monk, 1992a, 1992b; White and Michelmore, 1996). Others show that calculus students are unfamiliar with graphs of families of functions and relationships between equations and their graphs (Hauger, 1991; Evan, 1993). Still others concern precalculus or calculus students' knowledge of functions (Dreyfus and Eisenberg, 1982, 1983, 1987; Ayers, Davis, Dubinsky, and Lewin, 1988; Vinner, 1983; Vinner and Dreyfus, 1989). All of these studies deal with knowledge traditionally required to study calculus and which we who teach calculus often assume our students have.

Other studies deal more directly with specific calculus concepts and chronicle difficulties students have with these concepts. Some of these studies deal with calculus students' knowledge of integration (Orton, 1983b), limits (Cornu, 1981; Tall and Vinner, 1981; Monaghan, 1991; Williams, 1991; Confrey, 1980; Graham and Ferrini-Mundy, 1989; Tall and Schwarzenberger, 1978; Davis and Vinner, 1986), and continuity of functions (Tall and Vinner, 1981; Karplus, 1979; Vinner, 1987). Knowledge of the concepts of limit and continuity are essential for the study of calculus. In as much as students struggle with these concepts, it is safe to conclude they may very well struggle with concepts based on an understanding of limits and continuity, concepts like the derivative and rate of change.

Most pertinent to this study are studies concerning students' attempts to make sense of rate of change. Many studies deal with calculus and postcalculus students understanding of rate of change. Orton (1983a, 1984) studied responses of a large group of calculus and postcalculus students to a variety of rate of change tasks. Many of these students could not correctly work standard elementary calculus problems involving rate of change. Monk (1992a, 1992b) asked calculus and postcalculus students to work two related rates problems. Almost none of them were able to work them in standard ways taught in elementary calculus.

In a study of advanced mathematics college students, Thompson (1994a) noted that these students could not provide a correct description of the function $f_h(x) = (f(x+h) - f(x))/h$ nor say what function was obtained by taking the limit of this family of functions as h goes to zero. These and other studies show that calculus and postcalculus students have difficulty working standard calculus problems involving rate of change and have weak understandings of the theoretical underpinnings of the derivative and rate of change.

Part of the story of learning calculus is about difficulties students have with rate of change. But another part of the story concerns the resources students bring to calculus that might support their learning of rate of change. One of the goals of the larger study from which this report comes was to examine knowledge that precalculus students have and which could be used in their construction of knowledge of rate of change in calculus. Part of that examination included looking at what these students did correctly and thinking about how that knowledge might support their learning of calculus. But I was also interested in seeing what errors these students made and, in the case where they discovered and corrected their errors, how those discoveries came about and led to refining their knowledge of rate of change. That is the main story of this paper.

Relevant Literature

Work at TERC. Several studies deal with knowledge of rate of change of precalculus students. Researchers at TERC (Technical Education Research Centers) used a device with a car moving on a track so that both the position and velocity of the car at each moment in time could be graphed. Students were shown a velocity graph and asked to predict a corresponding position graph. Then they moved the car on the track to try to produce the given velocity graph and compared the generated position graph with their predicted position graph. A main result of these studies (Nemirovsky and Rubin, 1991, 1992; Rubin and Nemirovsky, 1991; Monk and Nemirovsky, 1992) was that students assumed that the position graph should resemble the velocity graph. If the velocity graph increased, these students believed the

position graph did too. If the velocity graph decreased, then for these students so did the position graph. An increasing velocity graph that lies below the horizontal axis yields a decreasing position graph and a decreasing velocity graph that lies above the horizontal axis yields an increasing position graph.

Students also predicted position graphs that had the same shape as the velocity graphs. If the velocity graph was a straight line, then they predicted the position graph would also be a straight line. If the velocity graph curved upward, they predicted that the position graph would too. Generally velocity and position graphs do not have the same shape although there is a relationship between the shapes of the two graphs. If the velocity graph is a horizontal line, then the position graph will be a straight line, though it will not be horizontal. It will slant up if the velocity graph is above the horizontal axis and slant down if the velocity graph is below the horizontal axis. But if the velocity graph is a straight line above the horizontal axis and slanted up, then the position graph will be curved upward.

For these students any change in the velocity graph meant the same kind of change in the position graph. If the velocity graph changed from increasing to decreasing, then these students predicted position graphs that also changed from increasing to decreasing. If the velocity graph changed from positive to negative, then according to these students so did the position graph. If a velocity graph is above the horizontal axis and changes from increasing to decreasing at a particular time value t_1, then the position graph is increasing the whole time but at t_1 it changes from increasing at an increasing rate to increasing at a decreasing rate. When a velocity graph changes from positive to negative, the position function changes from increasing to decreasing.

<u>One student's understanding of the sign of the velocity</u>. These studies show difficulties students have in making sense of rate of change. In an interview with one of these students (Laura), Nemirovsky (1994) reported her discovery of errors in interpreting the sign of the velocity and coming to a new understanding of what that sign means. Initially Laura

associated the sign of the velocity with speed of the car - positive velocity means going slower and negative velocity means going faster. In this context zero velocity meant changing from going faster to going slower or from going slower to going faster. After several trials with moving the car on the track, Laura came to understand the sign of the velocity as indicating distance from the motion detector - positive velocity means farther away from the motion detector and negative velocity means closer.

After suggestions from the interviewer, Laura moved the car in particular ways and came to see the sign of the velocity in terms of direction of motion in reference to the motion detector - positive velocity means moving away from the motion detector and negative velocity means moving toward the motion detector. Laura's ideas about the sign of the velocity changed as a result of experiences with the device. Initially her predictions did not match the graphs the device produced. As she worked with the device and the graphs it produced, she formed new hypotheses about the meaning of the sign of the velocity until the device produced graphs which matched those hypotheses.

<u>Work by the Thompsons</u>. Patrick and Alba Thompson (1992) reported on interviews with middle school students around situations involving constant rate. Thompson (1994b) gave a detailed account of one of these students as she came to understand constant rate of change. Initially she thought about speed as a distance, how far an object moves in one unit of time. For her, a given distance was measured in terms of units of speed-distance to yield the amount of time it takes to travel that distance, with one speed-distance unit corresponding to one unit of time. A distance of 100 feet can be traveled in five seconds at a speed of 20 feet per second since we can lay down five units of the speed-distance "20 feet per second" to add up to 100 feet. Her view of rate made distance the primary focus with time simply going along "for the ride".

Over several sessions her understanding of rate was extended as she came to see that distance and time changed together so that the ratio of changes in distance to corresponding

changes in time were all the same constant as was the ratio of the total accumulated distance to the total accumulated time after an increment of each. Her experiences with speed during these sessions involved forming conjectures about speed and testing them with a computer model of speed. They led her to refine her understanding of rate.

These studies deal specifically with knowledge of rate of change in students who have not yet had calculus. Both of the cases of individual students given are relevant to this study since they show students refining their knowledge of rate by forming and testing conjectures until those conjectures match the results of their testing. In the study presented in this paper we see four precalculus students doing the same thing.

What does it mean to know rate of change?

Generally, rate of change refers to change in one varying quantity with respect to change in another varying quantity. More specifically, we think of rate of change in the context of a function describing a relationship between quantities with one quantity represented as the independent variable (and often labeled x) and the other represented as the dependent variable (and often labeled y). Then rate of change refers to change in the dependent variable with respect to change in the independent variable.

As an example, consider Task 1 in Appendix A. Here we have two varying quantities - time and the distance between the two people at each moment in time. A function relating these two varying quantities has time as the independent variable and the distance between the two people as the dependent variable. The rate of change of this function refers to change in the distance between the two people with respect to change in time. The domain of a function is the set of allowable values of the independent variable and the range is the set of values of the dependent variable. In Task 1, the domain consists of time values from zero to eight seconds and the range is the set of numbers from zero to twenty feet.

Functions as graphs. Functions can be represented in several ways. We will concern ourselves here with functions represented as graphs and tables of values. Graphs are usually

constructed with the independent variable labeled and scaled on the horizontal axis and the dependent variable on the vertical axis. Each point on the graph of a function corresponds to a particular value of the independent variable and the value of the dependent variable associated with it. In Task 1, a graph of a function for the relationship between time and the distance between the two people would contain the point (0,20). This point would mean that at zero seconds, the distance between the two people was 20 feet. A graph of a function is all points formed by a value of the independent variable and its associated dependent variable value and positioned on a grid with horizontal and vertical axes. (See Appendix C for several correct graphs for Task 1.)

Functions as tables of values. A function can also be represented by a table of values which frequently are constructed using two columns with one column representing values of the independent variable and the other column representing the corresponding values of the dependent variable. For Task 1, a table could be constructed so that the left column contains values of the independent variable time and the right column contains values of the dependent variable distance between the two people. The left column of the table would contain zero seconds with a corresponding value of 20 feet in the right column. This would mean that at zero seconds the distance between the two people was 20 feet. A table of values for many functions cannot contain entries for every value of the independent variable since there are an infinite number of values that the independent variable can have. Instead a table will have entries for selected values of the independent variable with the corresponding values of the dependent variable. (See Appendix C for several correct tables for Task 1.)

Rate of change and graphs. Graphs of functions can reveal information about rate of change of the dependent variable with respect to the dependent variable. Appendix B shows seven shapes of graphs (Nemirovsky, 1991). Graphs of most continuous functions can be formed one of these shapes or by combinations of two or more. If a graph is a straight line, then the change in the dependent variable for every one unit change in the independent

variable is the same. If that line is a horizontal line as in Graph 1, then for every one unit change in the independent variable, there is no change in the dependent variable. If that line has a positive slope, say its slope is 2 as in Graph 2, then every one unit change in the independent variable means the dependent variable increases two units. If that line has a negative slope, say its slope is -3 as in Graph 3, then for every one unit change in the independent variable, the dependent variable decreases three units.

Graph 4 represents a function that is increasing and concave upward. It shows that each successive one unit increase in values of the independent variable is accompanied by successively larger changes in the value of the dependent variable. Graph 5 represents a function that is increasing and concave downward. It shows that each successive one unit increase in the independent variable is accompanied by successively smaller changes in the dependent variable. For both of these graphs, the dependent variable is increasing but in Graph 4 the dependent variable is increasing more and more and in Graph 5 the dependent variable is increasing less and less as the independent variable increases in one unit steps.

Graph 6 represents a function that is decreasing and concave upward which means that each successive one unit increase in the independent variable means successively smaller changes in the dependent variable. Graph 7 represents a function that is decreasing and concave downward which means that each successive one unit increase in the independent variable means successively larger changes in the dependent variable. For both of these graphs, the dependent variable is decreasing but in Graph 6 the dependent variable is decreasing less and less and in Graph 7 the dependent variable is decreasing more and more as the independent variable increases in one unit steps.

Rate of change and tables. Tables of values for functions can yield information about the rate of change of the dependent variable with respect to the independent variable. Appendix B shows tables of values for each of the seven basic shapes. For each table the values of the independent variable are in the left column and increase in one unit steps;

corresponding values of the dependent variable are in the right column. In Table 1, the values of the dependent variable do not change at all. In Table 2, there is a two unit increase in the dependent variable for every one unit increase in the independent variable. In Table 3, each one unit increase in the independent variable is associated with a three unit decrease in the dependent variable. These tables go with Graphs 1, 2, and 3.

For Table 4, the values of the dependent variable increase by successively bigger steps. For Table 5, the values of the dependent variable increase by successively smaller steps. In both tables, the values in the right column are increasing but in Table 4 they are increasing more and more and in Table 5 they are increasing less and less. These tables go with Graphs 4 and 5.

In Table 6, the values of the dependent variable decrease by successively smaller amounts. In Table 7, the values of the dependent variable decrease by successively bigger amounts. In both tables, the values in the right column are decreasing but in Table 6 they are decreasing less and less and in Table 7 they are decreasing more and more. These tables to with Graphs 6 and 7.

<u>Rate of change in Task 1</u>. The situation in Task 1 involves two people starting at opposite corners of a room and walking toward each other, slowing down as they get closer to each other, passing, and then speeding up as they get farther away from each other. This means that as they walk toward each other, the distance between them decreases and after they pass and walk away from each other, the distance between them increases. Since they slow down as they walk toward each other, the distance between them decreases less and less each passing second and since they speed up after they pass and walk away from each other, the distance between them increases more and more each passing second.

Therefore the graph of the function relating time and distance between the two people will have the same shapes as Graph 6 from zero seconds to the time they meet, say at $t=4$, and Graph 4 from $t=4$ to eight seconds. (In Task 1, the two people do not necessarily have to

meet at t=4 but they can meet at t=4. Since all four of the precalculus students who are the subjects of this paper had them meet at t=4 we will also have them meet at t=4 also in this discussion.) Figure 1 shows such a graph. Similarly, Figure 2 shows a table of values for this graph. Notice that for time values from zero to four, the distance values in the table decrease by successively smaller steps and for time values from four to eight the distance values increase by successively larger steps. The top half of this table looks like Table 6 and the bottom half looks like Table 4.

Rate of change in Task 2. In Task 2 the two people are walking the same steady pace the whole way. As in Task 1, the distance between them decreases as they walk toward each other and then increases after they pass and walk away from each other. But unlike Task 1, in Task 2, since the two people are walking the same pace the whole way, the distance between them decreases the same amount each second until they meet and increases the same amount each second as they walk away from each other.

Therefore the graph of the function relating the time and the distance between the two people will have the same shapes as Graph 3 from zero seconds until four seconds and Graph 2 from four seconds until eight seconds. Figure 3 shows this graph. Similarly Figure 4 shows the table of values for this graph. For time values from zero to four seconds, the distance between the two people decreases five feet each second and for time times from four to eight seconds, the distance between the two people increases five feet each second. The top half of this table resembles Table 3 and the bottom half resembles Table 2.

Figure 1

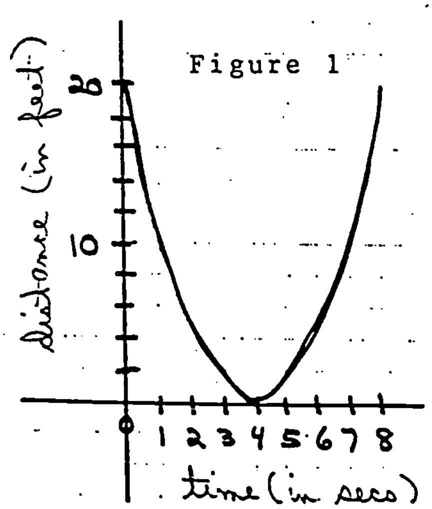

Figure 2

Time	Distance
0 seconds	20 feet
1 second	10 feet
2 seconds	5 feet
3 seconds	2 feet
4 seconds	0 feet
5 seconds	2 feet
6 seconds	5 feet
7 seconds	10 feet
8 seconds	20 feet

Figure 3

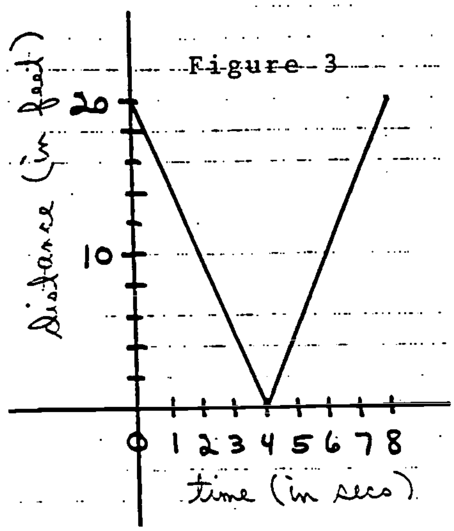

Figure 4

Time	Distance
0 seconds	20 feet
1 second	15 feet
2 seconds	10 feet
3 seconds	5 feet
4 seconds	0 feet
5 seconds	5 feet
6 seconds	10 feet
7 seconds	15 feet
8 seconds	20 feet

Methods

Subjects. The larger study from which this report comes included 37 students; 12 precalculus, 15 calculus, and 10 postcalculus. The precalculus students were high school students attending the same school and taking a junior-senior level class called precalculus from the same teacher. This group consisted of six males and six females. The four students whose responses are reported here were from this group of 12 students. Three of them were female and one was male.

Data collection. Students were interviewed individually and paid for their time. The interviews were audio and video taped. Each interview lasted about an hour and consisted of six tasks. In the first three tasks students were asked to make a graph and a table of values showing the distance between two people as they start from opposite corners of a room and walk toward each other, speed up, slow down, or maintain the same steady pace as they get closer to each other, pass, and then slow down, speed up, or maintain the same steady pace as they further apart, respectively. It is primarily the first task that is the concern of this paper.

For the fourth task, students were given a graph and a table of values for the relationship between time and the height of a ball thrown straight up into the air. They were asked to describe the height of the ball over the time, identify intervals of time when the ball traveled more and less feet, and find the speed of the ball at several points in time. In the fifth

task students were given a graph and table of values for the growth of a biomass over several hours. Students were asked to describe the growth of the biomass over time, identify intervals of the time when the biomass grew more and less, and find the rate of growth of the biomass at a particular point in time. Students were then given a more detailed graph and table of values around this point in time and asked to get, if they could, a better answer for the rate of growth of the biomass at that point in time. Students' responses to Tasks 4 and 5 have been discussed elsewhere (Hauger, 1995, 1996).

The sixth task showed a graph and table of values for the relationship between the radius and area of circles formed when a pebble is dropped into a calm pond. Again students were asked to describe the area as the radius increased, identify intervals of the radius for which the area was growing more and less, and find the rate of change of the area for a particular value of the radius.

Data analysis. The video and audio tapes were transcribed. In the larger study the responses of students were then analyzed in two ways. The first way involved categorizing strategies students used to answer each item, deciding types of responses indicating each category, and placing students' responses in these categories. The second way involved looking for patterns within students/across items and across students/within items. Of particular interest were differences in the distribution of frequency of use of strategies among the three groups of students and the reasoning given by students for their responses.

The part of the larger study reported here focuses on the reasoning of four precalculus students who initially drew the same incorrect graph for Task 1 and how they came to discover and correct their error. The particular data used in the analysis of these four students' responses are the graphs they made, the actual words and gestures they used to describe their graph and explain why they thought it did or did not meet the requirements for Task 1, and their reasoning about the graph, Task 1, and, for two of the students, Task 2 that helped them see their error and correct it.

Results

Students were given the following task:

> Task 1 - Two people start at opposite corners of a room and walk toward each other. As they walk, they both slow down as they get closer to each other, pass, and then speed up as they get farther apart. This takes a total of eight seconds. The opposite corners of the room are 20 feet apart.

Students were asked to sketch a graph showing the relationship between time and the distance between the two people and to make a table of values for their graph and explain how each showed the two people slowing down and speeding up. Appendix A gives the exact questions asked of students and templates for the graph and table of values. Appendix A also gives the situation for Tasks 2 and 3. Task 2 is relevant to the responses of two of the four students.

We have already discussed the basic shape of a correct graph and table of values for Tasks 1 and 2. (See Figures 1, 2, 3, and 4 on pages 11 and 12.) Nearly all of the 37 students in the larger study ended up with correct graphs on Tasks 1, 2, and 3 although not necessarily on the first try. Only two students, one precalculus and one calculus student, gave final incorrect graphs on at least two of the three tasks but even then their graphs correctly portrayed the rate of change.

On Task 1 almost half of the students (17 of 37) drew initial graphs that were incorrect. Of these 17 students six realized their error immediately after drawing their graph; seven more realized their error when they were asked to explain how their graphs showed the two people slowing down and then speeding up; another student realized her error when she made a table of values for her graph and tried to explain how it showed the two people slowing down; and one more student realized her error when she began working with Task 2. Two of these 17 students never realized their error.

Of the 17 students who drew incorrect initial graphs for Task 1, eight had graphs that were appropriate for Task 2. What would explain this? Almost all students quickly (and correctly) recognized that points (0,20) and (8,20), and some point near (4,0) should be on their graph. The simplest graph containing these three points is two straight line segments. Hence it is not surprising that over 20 percent of the students gave this as their first graph even though it was not appropriate for Task 1. They may have drawn this graph because they really thought it was a correct graph for Task 1. Or they may have wondered if it was a correct graph and just drew it so they could have a look at it to see if it had the right rate of change. Or they may have realized from the start it was not a correct graph for Task 1 but by getting it on paper, they could see what kinds of properties it had and then think about what kind of graph would meet the requirements for Task 1.

Some of these eight students knew fairly immediately that their graph was not right but it is not clear they knew this before they actually drew the graph. This makes me think that drawing the simplest graph they knew helped them think about the task and its rate of change. The following quotes are quite lengthy but show the full range of responses in terms of when students realized they had made an error (interviewer's dialogue is in parentheses):

> "[draws v-shaped graph] Uh. I think I did it wrong. Uh. Um. As they're - actually they're getting - ok. They're 20 feet apart when they start and then towards the middle of the time they'd be passing each other and then as the last four seconds progressed they'd be going towards the opposite sides of the room. (You said something about you weren't sure you did it right.) Right. But I can't decide whether it's right or not. (Ok. Well as we try to answer these questions, maybe that will clue you. How does the graph show the two people slowing down?) Right. See that's what I - but I don't know how it - [we read the problem again together]. So it'd come down and curve and then - it wouldn't be pointed [she points to the middle bottom of her graph which is pointed at t=4]. It would be a curve because they're - because they're taking - yeah, because they're taking the time and then they're gradually [traces pen along an imagined curve]. It'd be steeper up here [upper left of graph] and then like the slope wouldn't be as much down here [between t=2 and t=4]. [She draws a new graph shaped like a parabola, a graph appropriate for Task 1.]" MH - precalculus

"[draws v-shaped graph] (How does your graph show the two people slowing down?) It doesn't show. (What does your graph show?) It shows the distance between two people at different time. (And what does it show about speed? You said it didn't show them slowing down. Why doesn't it show them slowing down?) The distance and time - the speed is the same because it doesn't go in a curve. (So it's showing that they are going the same speed the whole time.) Yeah. [we go on to Task 2 and then return after Task 2 to Task 1.] (But here the graph is supposed to show them slowing down as they get closer to each other, passing, and then speeding up as they get farther apart.) [draws a correct curve without hesitation]." SCL - precalculus

"If they're steadily slowing down and they're steadily speeding up, it should just be straight lines [she draws a v-shaped graph]. (How does your graph show the two people slowing down?) It shows that at say one second they would be approximately 16 feet apart and at two seconds they'd be approximately about 10 feet apart, three seconds they'd be about five feet apart. So they're steadily slowing down, approximately five feet per second and the same way back speeding up [running her pen along the right side of her graph]. [She makes a table of values for her graph.] (How does your table of values show the two people slowing down?) It shows the same way that for each second - wait a minute - ok - how does it show them slowing down? (Yeah. How does your table of values show the two people slowing down?) Hmm. It shows they're getting closer, but it doesn't (How does it show them getting closer?) Well, it shows that they're getting closer because as each second passes, they get about five feet closer. But if they were slowing down, then it really should be more like 16 - more like - they should be - well, if they're really slowing down, enough to be noticeable, then they should be crossing more distance per second when they're speeding up than when they're slowing down. So it shouldn't be so constant. (Yeah. So what would be a graph that shows that?) [She works carefully with distances on the graph to create a table of values that shows slowing down and then speeding up and then creates her graph from this table of values.]" LC - precalculus

"I'd say in a v. [She draws a v-shaped graph.] (How does your graph show the two people slowing down?) By the decrease in the line [running her pen down the left line segment of her graph]. (Ok. And how does your graph show the two people speeding up?) By the increase in the line [running her pen up the right line segment of her graph]. (Ok. You mean the line is going up?) Yeah. (Ok.) [She makes a table of values for her graph and gives the same kind of explanation for how it shows speeding up and slowing down. Then we move on to Task 2. After some struggle with this one, I guide her to see that the characteristics of this problem call for the graph she drew in Task 1.] (Every second they went down five feet - and then after that, every second they went up five feet. Now, is that characteristics of them slowing down and then speeding up or is that characteristic of them going the same speed every second?) Oh. So did I do the? Ok. So this graph [the graph she drew for Task 1.] should really be for the second one [task 2]? (Yeah.) Now I don't know what the first one [graph for Task 1] is. [I repeat the conditions of Task 1.] (So what kind of shape of a graph?) Maybe like a parabola type shape. [She goes on to give appropriate reasons by a parabola-shaped graph meets the conditions of Task 1.]" KG - precalculus

Of the eight students whose initial graphs in Task 1 were appropriate for Task 2, four were precalculus, two were calculus, and two were postcalculus. Six of the eight students

(two of the four precalculus, both calculus, and both postcalculus students) realized fairly immediately after drawing their graphs that it did not show the two people slowing down and then speeding up. This includes MH and SCL quoted above. MH realized her graph was wrong before I asked her the first question. SCL realized his graph was wrong when I asked him about how his graph showed the two people slowing down. He may have realized his graph was wrong before this but he gave no indication he did. One of the precalculus students (LC) realized her graph was wrong after she constructed her table of values and started to explain how her table showed the two people slowing down. The other precalculus student (KG) realized her error only after she began working on Task 2.

I have indicated why I think students were tempted to draw a v-shaped graph for Task 1. But how did they come to see their error? And how did this help them know how to construct a correct graph, which all eight of them eventually did? I will now look in detail at each of the four students' quoted above to try to form answers to these questions.

MH's graph. As soon as MH drew her graph she suspected it was wrong: "[Draws v-shaped graph] Uh. I think I did it wrong.". She then gave a general description of her graph in terms of the two people walking toward each other, passing, and moving to the opposite corners and expressed indecision about "whether it's right or not". I suggested that we proceed with the questions for this problem - maybe that would give her some clues. Then I asked her how her graph showed the two people slowing down:

> "Right. See that's what I - but I don't know how it - [we read the problem again together]. So it'd come down and curve and then - it wouldn't be pointed [she points to the middle of the bottom of her graph which is pointed at $t=4$]. It would be a curve because they're - because they're taking - yeah, because they're taking the time and then they're gradually [traces pen along an imagined curve]. It'd be steeper up here [upper left of graph] and then like the slope wouldn't be as much down here [between $t=2$ and $t=4$]. [She draws a new graph shaped like a parabola, a graph appropriate for Task 1.]" MH - Precalculus

At the beginning of this section of dialogue MH was still unsure how the graph should look. But after she and I read the problem statement again, she began to verbally construct an appropriate graph. "So it'd come down and curve ... because they're taking the time". She

recognized that the graph could not be straight; it had to curve to show the two people slowing down. A possible interpretation of "they're taking the time" is that as the graph comes down from (0,20) to (4,0), it curves because the time it takes to travel a certain number of feet is greater than it was when they first started walking. This makes time a salient factor in that part of the graph.

MH followed these comments with more specific comments about the slope the curve needed to have in order to show the two people slowing down as they get closer: "It'd be steeper up here [upper left of graph] and then like the slope wouldn't be as much down here [between t=2 and t=4]." She then drew an appropriate graph.

The sequence of events leading to MH's construction of a correct graph began with a recognition that the graph could not be straight - therefore it had to be curved. MH then used this curved aspect to make a connection between time and distance and taking more time to cover a given distance in the curved part of the graph. She then connected that to slope - the upper left section has steeper slope and the section from t=2 to t=4 does not have as much slope (the graph is flatter there).

These connections are important although it is not clear that MH explicitly saw them or that they represented a rational sequence of thought to her. It could be that her talk was random mumbling as she tried to figure out what the graph ought to look like. However, since her words make sense in terms of properties of her final graph and since they make sense in terms of understandable connections among speed, slope, and relationship between distance and time, I will discuss them further here.

If straight means constant speed, then curved means changing speed which means more or less time to cover the same distance. This is one of several ways students describe changing speed. Slowing down means more time to cover the same distance. So that means less time to cover the same distance on the first two seconds (since they were going faster there) and more time to cover the same distance from t=2 to t=4 (since they were slowing

down there). Less time to cover a given distance means larger slope (fixed delta d over small delta t) and more time to cover the same distance means smaller slope (fixed delta d over large delta t). So that means steeper slope in the upper left part of the graph and slope that "wouldn't be as much" between t=2 and t=4. This is a picture of what this student may have been thinking that helped her create a correct graph for Task 1. For the rest of Tasks 1, 2, and 3, MH used slope (steeper, more gradual, same slope) and distance per second to talk about speeding up, slowing down, and moving at a steady pace.

SCL's graph. When I asked SCL how his graph showed the two people slowing down, he said "It doesn't show ... the speed is the same because it doesn't go in a curve." Like MH, SCL realized that curved graph means changing speed and since he did not have a curve, his graph did not show slowing down. In the interview we went immediately on to Task 2 where he identified the graph he drew for Task 1 as the appropriate graph for Task 2. He justified his graph with:

> " 'Cause they take same time going a certain distance. (So in the first second they're going how far?) About five feet. (And in the second second?) About five feet. Five feet each second." SCL - Precalculus

SCL used the same relationship between distance and time that MH used - amount of time to go a certain distance. MH used this relationship to talk about slowing down - taking more time to go a certain distance - while SCL used this relationship to talk about going the same steady pace - taking the same time to go a certain distance. I oriented SCL away from "number of seconds per feet" to "number of feet per second"; he picked up on it readily by describing the rate of change of his graph in terms of "five feet each second".

SCL completed a table of values for Task 2 without looking at the graph. He did not need to since he knew that each second the graph changed five feet. But when I asked him how his graph showed the two people walking at a steady pace, he said:

> "When they get there - for four seconds they meet. So in four seconds they go same distance going across room in same time as they come to this point [pointing to (4,0)]. (So the fact that it took them fours seconds to meet and four seconds to get to the other side, that shows that they were going at a steady pace?) Uh um." SCL - Precalculus

SCL responded that his table showed the two people walking at a steady pace since they were covering a total of 20 feet per four seconds on the first half and 20 feet per four seconds on the second half. For each of Tasks 1, 2, and 3, it is possible to get correct graphs and tables that have changes of 20 feet per four seconds on the first half and 20 feet per four seconds on the second half. (See Appendix C.) So this condition is not sufficient to show walking at a steady pace.

Earlier when explaining how his graph showed steady pace, SCL said " 'Cause they take same time going a certain distance.". I thought at the time he was referring to small distances and small amounts of time, like five feet and one second. But he could have been referring to 20 feet with the intent that since they covered 20 feet in four seconds on one side of the graph and 20 feet in four seconds on the other side of the graph, then that showed steady pace. This represents one of the difficulties of learning rate of change - steady pace means same number of feet in same amount of time. More precisely, steady pace means the ratio of distance covered to the time to cover it is the same no matter how short or long the interval of time. For Task 2, I considered answers of five feet per second as adequate evidence for steady pace. However, 20 feet per four seconds in not adequate evidence for steady pace.

SCL and I then returned to Task 1. After I repeated the statement of the task, SCL drew a correct curve without hesitation. When I asked him how his graph showed the two people slowing down he said:

"Here, the graph shows like in the first [runs finger along curve from (0,20) to (1,5)] as the time goes, close to the meeting point [runs finger along horizontal axis from t=0 to t=4], the distance gets smaller, decrease more greatly than from here to here [makes a motion with his hands indicating the distance moved between (0,20) to (1,5) on the graph for Task 1 is more than the distance moved between (0,20) to (1,5) on Task 2]. From here to here [(3,5) to (4,0) on the graph for Task 2] decrease more [compared to the decrease from (3,.5) to (4,0) on the graph for Task 1] than from here to here [gestures from (0,20) to (1,15) on the graph for Task 2]. (So you're saying that in the first second they went that far [running pen along graph for Task 1 between (0,20) and (1,5)] and the second second they went less?) Yeah. (And how does your graph show them speeding up?) Like from here [(4,0)] to here [(5,.1)], this much. From here [(5,.1)] to here [(6,1)], they go more. And then last time, they go [runs finger along the graph for Task 1 from (7,3) to (8,20)]." SCL - Precalculus

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In lines 1 through 8 of this dialogue SCL explained how his graph for Task 1 showed the two people slowing down. He did this by comparing distance per second on his Task 1 graph with distance per second on his Task 2 graph. In lines 1 through 5 he said that the distance on his Task 1 graph "decrease more greatly" the first second than on his Task 2 graph. His Task 1 graph went from (0,20) to (1,5) and his Task 2 graph went from (0,20) to (1,15) so the distances on graph 1 between $t=0$ and $t=1$ did indeed decrease "more greatly" than the distances on graph 2. And therefore, the two people were walking faster in the first second on graph 1 than on graph 2.

Then in lines 6 through 9 he said that the graph for Task 2 "decrease more" from (3,5) to (4,0) than the graph of Task 1 as it went from (3,.5) to (4,0). That is an indication that the two people were walking slower on graph 1 between $t=3$ and $t=4$ than they were on graph 2. I thought that was what SCL was meaning until he then went on to compare differences in distance moved between the two graphs from $t=3$ to $t=4$ to distance moved on graph 2 between $t=0$ and $t=1$. I finally came to interpret this to mean he was noticing that from $t=0$ to $t=1$, graph 1 decreased more than graph 2 but from $t=3$ to $t=4$ graph 2 decreased more than graph 1 which is a contrast with what is happening between $t=0$ and $t=1$. Therefore, I took his words "From here to here [between $t=3$ and $t=4$ on graph 2] decrease more [compared to between $t=3$ and $t=4$ on graph 1] than from here to here [(0,20) to (1,15) on graph 2]" to mean he was noticing that what was happening on the two graphs between $t=0$ to $t=1$ is "the opposite" of what was happening between $t=3$ to $t=4$.

Between $t=0$ and $t=1$, graph 1 decreased more and between $t=3$ and $t=4$, graph 2 decreased more. Since graph 2 decreased the same every second, this means that on graph 1, distance decreased more between $t=0$ and $t=1$ than it did between $t=3$ and $t=4$. Therefore on graph 1 the two people were covering more distance between $t=0$ and $t=1$ than between $t=3$ and $t=4$ and therefore they were slowing down. My follow-up question in lines 8 through 10 and his answer seem to verify that this is what he was thinking. And his answer to

my question about how his graph 1 showed the two people speeding up seemed to confirm this (lines 10 through 13). He clearly indicated that since the graph was increasing successively more each second from t=4 to t=8, then the two people were speeding up.

In summary, SCL recognized that his first graph for Task 1 was wrong because on that graph "the speed is the same because it doesn't go in a curve." To get a more appropriate graph, SCL drew a curved graph, actually the most obvious curved graph joining (0,20), (4,0), and (8,20), a parabola-like curve with vertex at (4,0). His description of graph 2 and his subsequent description of his correct graph 1 made me think he was focusing on distance covered in time segments. First he focused on time per distance. I then guided him toward distance per second; he turned to distance per four seconds. Then he tried to talk in terms of distance per second when comparing and contrasting graphs of Tasks 1 and 2. Finally he spoke clearly in terms of feet per second. For the remainder of Tasks 1 and 3 he described speed as feet per second.

<u>LC's graph</u>. Before LC drew her v-shaped graph for Task 1, she said "If they're steadily slowing down and they're steadily speeding up, it should just be straight lines.". When I asked her how her graph showed the two people slowing down, she focused on the fact that the graph was declining:

> "It shows that at say one second they would be approximately 16 feet apart and at two seconds they'd be approximately about 10 feet apart, three seconds they'd be about five feet apart. So they're steadily slowing down, approximately five feet per second and the same way back up [runner her pen along the right side of her graph]." LC - Precalculus

Since the graph was declining five feet per second, that meant, to LC, that the two people were slowing down and since the graph was increasing five feet per second after t=4, that meant that the two people were speeding up. I then asked her to make a table of values for her graph and to explain how her table showed the two people slowing down. It was here that she realized her graph was wrong:

> "It shows the same way that for each second - wait a minute - ok - how does it show them slowing down? (Yeah. How does your table of values show the two people slowing down?) Hmm. It shows they're getting closer, but it doesn't (How does it

show them getting closer?) Well, it shows that they're getting closer because as each second passes, they get about five feet closer. But if they were slowing down, then it really should be more like 16 - more like - they should be - well, if they're really slowing down, enough to be noticeable, then they should be crossing more distance per second when they're speeding up than when they're slowing down. So it shouldn't be so constant." LC - Precalculus

She started to give the same explanation she gave earlier - that the distance values in the table are declining every second - that shows slowing down - and then the distance values are increasing every second - that shows speeding up. But she stopped in midsentence as she realized that this did not show slowing down or speeding up - it shows them getting closer to each other and then getting farther apart (lines 3 to 5).

In lines 5 to 8 she began to form a statement about what it means to show slowing down - "They should be crossing more distance per second when they're speeding up than when they're slowing down.". She showed here that she understood that covering ("crossing" is the word she used) more distance each second meant speeding up and covering less distance each second meant slowing down. Actually her words have at least one other interpretation - that covering more distance per second means only that they are going faster, not necessarily that they are speeding up, and covering less distance means that they are going slower, not that they are slowing down.

This represents another subtle nuance in learning rate of change - distinguishing between going at a slow constant rate and going successively slower. The former means each second, the <u>same</u> **small** number of feet is covered; the latter means each second, fewer feet is covered than in the preceding second. We could have a relatively large number of feet covered each second with the number of feet covered each second less than the number of feet covered in the second before it - that would also mean slowing down.

In this dialogue sequence, LC also made one other comment worthy of note - "It shouldn't be so constant." (lines 8 and 9). This was right after she remarked that "They should be crossing more distance per second when they are speeding up than when they are slowing down." (lines 7 and 8). She had come to recognize that her graph did not show

slowing down and speeding up because it was changing a constant number of feet each second, the condition appropriate for Task 2. It was this realization here that made the Task 2 part of her interview short - a total of six lines of dialogue for the whole task - she already knew what the graph and table of values had to look like and why.

The next section of LC's dialogue shows how she arrived at a correct graph for Task 1:

> "(Yeah. So what would be a graph that shows that?) It would take them five seconds to meet because they're slowing down as they're reaching each other. But then once they pass each other, then they're speeding up so they would take less time. So it wouldn't be so balanced. And so then the times (Are you thinking about drawing a straight line here [from (0,20) to (5,0)] - instead of having it anchor here [at (4,0)] have it anchor there [at (5,0)]?)" LC - Precalculus

The first part of this dialogue segment illustrates a common way of thinking about slowing down and speeding up - since they are slowing down, it will take them longer to meet and since they are speeding up, it will not take them as long to get to the opposite corner. Actually this would show slower average speed over the first part of the graph (from t=0 to when they meet) and faster average speed over the rest of the graph. But it would not necessarily show slowing down and speeding up.

Appendix C shows a graph that is appropriate for Task 1 which meets the horizontal axis at (5,0). So it has a slower average speed from t=0 to t=5 (4 feet per second) and has a faster average speed from t=5 to t=8 (6 2/3 feet per second). But that is not what shows the graph "slowing down" from t=0 to t=5 and "speeding up" from t=5 to t=8. The graph shows slowing down from t=0 to t=5 since each second the height of the graph changes less than it did the previous second and speeding up from t=5 to t=8 since each second the height of the graph changes more than it did the previous second. On the same page is another graph that is appropriate for Task 1 but which meets the horizontal axis at (3,0) so that it has a faster average speed from t=0 to t=3 (6 2/3 feet per second) and a slower average speed from t=3 to t=8 (4 feet per second). That graph also shows slowing down from t=0 to t=3 (since each second the graph changes less than it did the second before) and speeding up from t=3 to t=8 (since each second it changes more than it did the second before).

Appendix C also shows a graph that is appropriate for Task 3 which meets the horizontal axis at (5,0). So it has a slower average speed from t=0 to t=5 and a faster average speed from t=5 to t=8. But the graph does not show slowing down from t=0 to t=5 and speeding up from t=5 to t=8. It shows speeding up from t=0 to t=5 (since the graph changes more each second than it did the previous second) and slowing down from t=5 to t=8 (since the graph changes less each second than it did the previous second). On that same page is a graph that meets the horizontal axis at (3,0) so that its average speed from t=0 to t=3 is faster than its average speed from t=3 to t=8. It also shows speeding up from t=0 to t=3 and slowing down from t=3 to t=8.

So meeting after t=4 does not characterize slowing down first and then speeding up, and meeting before t=4 does not characterize speeding up first and then slowing down. It takes something else to characterize slowing down and speeding up. LC worked through this in the next section of dialogue:

> "Let's see. Ok, at the beginning they're 20 feet apart - that's for sure - and so at one second they would cross more distance than they do in two seconds. Then in each passing second, {indecipherable} - so - {indecipherable}. (They're going to cover more distance in each passing second or less distance?) More in the first second than in the following until they meet and then they'll cover more again." LC - Precalculus

LC returned to her earlier description of crossing successively more or less distance to characterize speeding up or slowing down. They cover successively less distance each second until they meet and then they cover successively more distance each second after they meet. LC carefully constructed a table of values showing this and then plotted the points from the table to get her graph. She then made an observation:

> "Then it's not going to be a straight line [pointing to the left side of her graph]. So it's just going to be more of a curve than it's going to be a straight line. It's not a constant rate of change." LC - Precalculus

I find this dialogue interesting for two reasons. The first has to do with connections between distance covered each second and shape of graph and the second has to do with shape of graph and statement about rate of change.

Concerning connections between distance covered each second and shape of graph, LC already knew that the distance covered each second was not the same. It was less and less until the two people met and then it was more and more. But apparently she had not yet made a connection between a varying amount of distance covered each second and the shape of the graph. Nowhere in her interview to this point had she yet made mention of the shape of the graph - either curved or straight. In this dialogue segment, after she made an appropriate table of values showing slowing down and speeding up and made a graph from that table, she made her first comment about the shape of the graph - "Then it's not going to be a straight line." Until this point LC did not realize that covering different distances each second meant the graph would not be a straight line.

Making a connection between amount of change each second and the shape of the graph is both important and nontrivial. Corresponding connections between the graph of a function and its derivative is a major component of elementary calculus. Approximately one sixth of instruction time in the first semester of calculus is devoted to this connection, as judged by the placement and length of the corresponding chapter in most elementary calculus texts. LC had not yet made this connection - at least not until she actually drew a graph that showed the distances covered in each second.

The second reason this segment of dialogue is interesting is that once LC realized that the graph was not a straight line, she almost immediately connected that to a nonconstant rate of change - "It's not a constant rate of change.". From my point of view, knowing that the distances covered each second are not the same means that the rate of change is not constant. But it seems that this is not how LC arrived at this conclusion. It seems that LC made the conclusion about a nonconstant rate of change through the shape of the graph. The graph was not a straight line; a straight line means constant rate of change; therefore, "It's not a constant rate of change.".

One might wonder how students in a precalculus course would make a connection between lines and constant rate of change but not make a connection between curves and varying amounts of distance covered each second. One prominent topic in high school mathematics is lines and their properties. One of the important properties of lines is slope, which is constant for all pairs of points along the line and which is often called the rate of change of the line. Therefore, lines have constant rate of change. Contrast this with the lack of instruction in high school mathematics concerning certain properties of curves, like varying amounts of change in the dependent variable for constant changes in the independent variable. In this context, it is not surprising to me that LC knew that straight line meant constant rate of change (and therefore curve meant nonconstant rate of change) but did not know that varying amounts of distance covered each second did not necessarily by itself mean the graph was a curve.

After LC constructed her graph from her table of values, I asked her how her graph and table showed the two people slowing down and speeding up:

> "(How does your new graph show the two people slowing down?) It shows that they're slowing down because they are crossing less distance with each second. (And how does it show them speeding up?) Crossing more distance with each second. (Now how does your table of values show them slowing?) Because this shows that with each second [from t=0 to t=5] they're crossing less distance with each second and then they're crossing more distance with each second in these seconds [from t=5 to t=8, showing speeding up]." LC - Precalculus

LC made clear statements about why her graph and table of values showed slowing down and speeding up - they cross less distance with each second and they cross more distance with each second, respectively. Her graph still met the horizontal axis at t=5 but from t=0 to t=5 it correctly showed slowing down and from t=5 to t=8 it correctly showed speeding up.

In summary, LC originally thought about speeding up and slowing down in terms of position on the graph rather than distance covered each second (first dialogue segment). But when she worked with the table of values, she realized she should be attending to distance covered each second (second dialogue segment). In the third dialogue segment, she focused on

covering 20 feet in five seconds as evidence of slowing down and covering 20 feet in three seconds as evidence of speeding up. In dialogue segment four she returned to distance covered each second as a characterization of speed with covering successively less distance associated with slowing down and covering successively more distance associated with speeding up. Dialogue segment five shows her making connections between curve and varying distances covered each second. Finally, she gave a firm statement in terms of distance covered each second about what it means to slow down and speed up (dialogue segment 6).

For the remainder of her work on Tasks 2 and 3, LC used distance "crossed" each second to show steady pace, speeding up, and slowing down. On Task 3 she once again focused briefly on where they should meet. Since they were speeding up before they met and slowing down after, she felt it should take them less time to meet than it would for them to get to the opposite corner after they met. Therefore they must meet before t=4. In fact, she had them meet at t=2 but, more importantly, she positioned the points on her graph so that between t=0 and t=2, the graph does in fact show speeding up (since it "crosses" more distance the second second than the first second) and between t=2 and t=8 the graph shows slowing down (since it "crosses" less distance each second).

KG's graph. Like MH, SCL and LC, KG drew a v-shaped curve for Task 1. Unlike MH, SCL and LC, KG did not realize during Task 1 that this graph was incorrect. It was while she was working on Task 2, and with much guidance from me, that she realized her graph for Task 1 was appropriate for Task 2. How did she think about her incorrect graph for Task 1 and how did she come to see that graph as appropriate for Task 2?

She justified her graph for Task 1 in terms of position of the graph at each point in time rather than distance covered each second:

> "I'd say in a v. [She draws a v-shaped graph.] (How does your graph show the two people slowing down?) By the decrease in the line [running her pen down the left line segment of her graph]. (Ok. And how does your graph show that two people speeding up?) By the increase in the line [running her pen up the right line segment of her graph.] (Ok. You mean the line is going up?) Yeah. (Ok.)" KG - Precalculus

For KG, a decreasing graphs shows "slowing down" and an increasing graph shows "speeding up". Her graph showed neither "slowing down" or "speeding up" since it was two straight line segments. The first line segment showed the distance between the two people decreasing at a constant rate and the second line segment showed the distance between the two people increasing at a constant rate.

KG gave the same kind of description for her table of values:

> "[Her table of values has these d values: 20 15 10 5 0 5 10 15 20.] (How does your table of values show the two people slowing down?) Slowing down. Didn't we do that question already? (Yeah. You were using the graph.) Oh. Ok. How does the table? Just the numbers go down [pointing to the part of the table corresponding to t=0, 1, 2, 3, and 4] (You mean from here to here to here [pointing to 20 to 15 to 10]?) Yeah. (And how does your table show the two people speeding up?) 'Cause the numbers go up [pointing to the part of the table corresponding to t=5, 6, 7, and 8]." KG - Precalculus

For KG, decreasing values of d for increasing values of t shows slowing down and increasing values of d for increasing values of t shows speeding up. Like her graph, her table of values showed the distance between the two people decreasing at a constant rate and then increasing at a constant rate.

When she got to Task 2, she had difficulty thinking about how the graph should be:

> "They're going to - they're gonna meet up - and pass. (They're going to meet and pass, yeah. Again, they're going to meet and pass - just like they did here [pointing to the statement of Task 1].) Right. (They're going to walk, meet, pass, go to the other side.) But they're not going to slow down? (They're not going to slow down or speed up. They're both going to have the same pace the whole way.) Ok. They're both starting at 20 feet apart - and - hmm. I can't picture it. (Maybe it would be helpful to make a table of values first and then make the graph from the table.) I don't know if that would help me or not. (If they're going the same speed, what is happening to the distance between them?) Ok. The distance is getting less but they're both going - I know it should go down somehow because the distance can't stay - I'm having trouble with the time. Is it going to take them eight seconds total? (Yes.) Ok. [long pause] Unless - the only thing I can think of is if I put a point here [at (8,0)] and draw them both [moves her hand along the straight line from (0,20) to (8,0)] like that. (So they would be starting at 20 feet apart [points to (20,0)] - you're talking about putting a point here [points to (8,0)]?) After eight seconds were over. (So they would be zero feet apart at the end of two seconds?) You mean at the end of eight seconds? (I mean at the end of eight seconds.) Oh. Ok. They're still going to pass halfway through. (Yeah.) So we still need the four [we still need the point at (4,0)]. How do I get?" KG - Precalculus

In this segment, KG struggled with creating the graph. Since she had already used the simplest graph for Task 1, she could not easily come up with another graph that went through (0,20), (4,0), and (8,20). In addition, the graph she needed for Task 2 had already been used for Task 1, and since Tasks 1 and 2 are obviously different, she could not use the same graph for Task 2. So she was in a quandary about what to do. I tried to help her by suggesting she make a table of values (lines 6 and 7) or think about the relationship between same speed and changes in distance each second (lines 8 and 9). But this did not seem to speak to KG. I finally intervened more pointedly with the following suggestion:

> "(Well, one thing we ought to think about. Look at the characteristics of this graph [pointing to the graph she made for Task 1]. Every second they went down five feet. And then after that, every second they went up five feet. Now, is that characteristic of them slowing down and then speeding up or is that characteristic of them going the same speed every second?) Oh. So did I do the? Ok. So this graph [pointing to the graph she drew for Task 1] should really be for the second one [Task 2]? (Yeah.) Now I don't know what the first one is." KG - Precalculus

My comments about the graph decreasing five feet every second and then increasing five feet every second helped KG see that her graph for Task 1 was really appropriate for Task 2. But then she was unsure about an appropriate graph for Task 1. I suggested to her that we continue with Task 2 and then return later to Task 1:

> "(Ok. So how does your graph show the steady pace of the two people?) For every five feet we go, uh, what is it? It takes two, one second. (And now this table of values [pointing to the table made for this graph in Task 1], this is the table of values for this graph that you've drawn for question [Task] 2 [the graph she initially drew for Task 1]. How does this table of values show the steady pace of the two people?) Only that it's a certain interval, the numbers, it's always for every second, it goes five down and five up." KG - Precalculus

To explain how her graph showed steady pace, KG talked in terms of time it took to cover a fixed number of feet - "For every five feet we go, uh, what is it? It takes two, one second.". I puzzled over why she said two seconds before saying one second. In eight seconds, each person had to go 20 feet; so it took each person two seconds to go five feet. That may be why she said two seconds first. She then said one second, perhaps because she realized that with two people walking, the time it took for them to walk a total of five feet was

one second. Or perhaps she saw that the graph dropped five feet every second, a visual hint about how much time it took to move five feet. Her response to my question about how her table of values showed the steady pace of the two people made me think she was focusing on change in the d values on the graph and table: "it's always for every second, it goes five down and five up.". Each second the d values on the graph and in the table decrease by five and then increase by five.

We then returned to Task 1. I repeated the statement of the task before I asked her what shape graph would match that condition:

> "Maybe like a parabola type shape. [makes a parabolic sweep with her pen] (Ok.) Since they're slowing down it's going to take more seconds when they meet, er, when they're getting closer to meeting. But I'm not sure exactly where that would be. (I think there are many possibilities.) Ok. So it doesn't have to be (No. There's not going to be just one parabola shaped thing.) So should I just draw what I <u>think</u> is a possibility? (Yeah.) Ok. So the fewer feet there are, it's going to take them more seconds 'cause they're walking slower [running her pen along an imaginary curve between t=2 and t=4]. So that means [draws left side of graph and then right side yielding a curve that is concave upward from (0,20) to (4,0) to (8,20)]." KG - Precalculus

If a line is the most familiar graph for precalculus students, the second most familiar graph is a parabola. That makes parabolas a tempting graph after one has already used a line for a different situation. So her comment "Maybe like a parabola type shape." may have been a guess rather than a decision based on an analysis of the rate of change.

She then noted that "Since they're slowing down it's going to take more seconds when they meet, er, when they're getting closer to meeting.". There are two possible interpretations for this sentence. One is that as the two people get closer to meeting, it takes more time to cover the same amount of distance they were covering earlier. A second interpretation is that since the two people are slowing down, it takes longer for them to meet then it does for them to proceed to the opposite corner after they meet. In that case they would meet after t=4. This is similar to what LC did in her problem.

Her next comment - "But I'm not sure where that would be." - made me think she was really not sure where she wanted to put the meeting place - before t=4, at t=4, or after t=4 -

and therefore she was not thinking that they necessarily had to meet after $t=4$ and hence interpretation two above was not what she meant. After a brief discussion about there being several possibilities for the graph, she went on to talk about "walking slower" in terms of "taking more seconds" (lines 6 to 8). This made me think that the first interpretation above was the correct interpretation of her words, that slowing down, for her, meant taking more time to cover a given distance. She went on to draw a correct graph and that graph met the horizontal axis at $t=4$. I took this as further confirmation that the second interpretation above was not what she meant, that she did not mean that slowing down necessitates taking longer to meet.

When I asked her how her new graph for problem 1 showed the two people slowing down, she said:

> "As they're getting closer, it takes more seconds than just one second in time to travel a certain number of feet. (So, for example, in the first second, how far?) Ok. Oh. [points to the point (1,8)] (And then as you get closer - ok - so they went 12 and then in the second second they went down to three.) So they only went about nine feet. (Right. Then the last second they went a half a foot. Ok.)" KG - Precalculus

She justified her graph in terms of more time to cover a certain distance (time per distance - lines 1 to 2), further confirmation that the first interpretation above is the correct one. I tried to orient her toward "distance per time" in lines 2 to 3. Because I was not very adept at getting her to respond verbally, it is not clear she took to that orientation. When I asked her how far the two people had traveled in the first second, she responded by pointing to the point (1,8) which gives the height at $t=1$. This could mean that she knew that together the two people had traveled 12 feet in the first second. Or it could mean that she thought the point (1,8) meant they had traveled eight feet in the first second. Or it could mean that she did not know what to say but knew I was asking something about what was happening at $t=1$ and therefore simply pointed to the point (1,8).

I responded by saying how many feet they went in the first second - 12 - and then noted that at the end of the second second they were at $d=3$. At this point she seemed to have

interpreted my words to mean that they had gone from a d value of 12 feet at the end of the first second to a d value of 3 at the end of the second second and therefore "they only went about nine feet". Her phraseology here made me think she was thinking this: "They went 12 feet in the first second and 9 feet in the second second. That shows that they were slowing down, which confirms what I said before.". Why I responded with "Right." is unclear to me. I may have been busy noticing that on her graph the two people moved two and a half feet in the third second and then a half a foot in the fourth second, which I noted with "Then the last second they went a half a foot.".

My efforts to orient her toward "feet per second" began to pay off, for when I asked her how her graph showed the two people speeding up, she responded with:

> "Ok. It's the same idea. In this second [from t=4 to t=5] they were only going less than half a foot. And then in the next second, from 6 to 7, they're going a few feet and then from 7 to 8 they're going a lot.

It is important to point out that there are two interpretations of these words. One interpretation is that they moved less than half a foot between t=4 and t=5 (from (4,0) to (5,.5)), then they moved more feet between t=6 and t=7 (approximately two and a half feet from (6,1.5) to (7,4)), and then they moved a lot of feet between t=7 and t=8 (about 16 feet from (7,16) to (8,20)). The other interpretation is that since the height at t=5 is one and a half feet, then that means they moved that far since t=4; since the height at t=7 is four feet, then they moved four feet in that second; and since the height is 20 at t=8, then they moved 20 feet in that second. KG's first attempt at Task 1 would make us lean toward this second interpretation. But in her later work with Task 2 and her most recent comments about Task 1 she used "time per distance" and "distance per time". That made me believe she was talking here about "distance per second" and not "height at time values".

She made a table of values consistent with her new graph for Task 1 and when I asked her how her table of values showed the two people slowing down, she said:

> "Ok. When they're closer to each other they travel fewer feet. (Ok. So for here [from t=0 to t=1] they went a big distance.) Right and then it gets smaller as they get closer." KG - Precalculus

Again we have a quandary concerning interpretation - does she mean height at each time value or does she mean change in distance between time values? For the same reasons as above, I believe she means the latter. Her answer to why her table of values shows the two people speeding up confirms this:

> "It takes, especially down here [pointing to the interval from t=7 to t=8 in the table of values], it takes fewer seconds to go a certain distance." KG - Precalculus

Here KG is clearly focusing on "time per distance" and not "height at time value". So I believe we can conclude that she had made a transition from considering height at time value, as in her initial attempt at Task 1, to a relationship between time and distance, as in her work with Task 2 and her return to Task 1. Her subsequent work on Task 3 showed her using "distance per time" exclusively, and rather articulately, as her measure of speed:

> "[draws a correct graph - concave downward from (0,20) to (4,0) and from (4,0) to (8,20)] (How does your graph show the two people speeding up?) Ok. When they're speeding up - say from two and half seconds to four seconds, they traveled quite a distance. But in the first two seconds [from t=0 to t=2] they only covered about five feet. (So they were going faster down here [lower part of left side of graph] and slower up here [upper part of left side of graph]?) Yeah. (And how does your graph show the two people slowing down?) Ok. The first two seconds [from t=4 to t=6], they were going pretty fast still. They covered about 15 feet. And then in the last two seconds [from t=6 to t=8] they only covered about five feet." KG - Precalculus

Here KG was clearly <u>not</u> talking about height as a measure of speed since she identified going slower with the part of the graph that had large height values [from t=0 to t=2 and from t=6 to t=8] and going faster with the part of the graph that had small height values [from t=2 to t=6]. The two people were speeding up from t=0 to t=4 since they were going slower from t=0 to t=2 and faster from t=2 to t=4. Then the two people were slowing down from t=4 to t=8 since they were going faster from t=4 to t=6 and slower from t=6 to t=8. Similarly she used "distance per time" to talk about speed with her table of values:

> "[Makes a table of values that goes with her graph] (How does your table of values show the two people speeding up?) The difference between per second is greater the closer you get to four [t=4]. (It's small here [from t=0 to t=1 to t=2].) Right. (And then it gets bigger here [from t=2 to t=3 to t=4]. How does your table of values show the two people slowing down?) The distance increases by quite a bit here

[from t=4 to t=5] and then by less as we go along [from t=5 to t=8]." KG - Precalculus

She talked here about the "difference between per second" (line 2) which I took to mean change in distance each second, which does indeed get greater the closer you get to t=4. She could not have been referring to height at time values since that does not get greater as t goes from 0 to 4.

Her use of the word "distance" (line 5) could mean either change in distance between values of t or height at t values; either interpretation yields a correct statement. If it means change in distance between values of t, then her statement "distance increases by quite a bit here" (line 5) refers to a large change in distance from t=4 to t=5, hence "going faster". Her statement "and then by less as we go along" (line 6) then refers to a small change in distance per second from t=5 to t=8, hence "going slower". So the two people are "slowing down".

If "distance" means height at each t value, then her statement "distance increases by quite a bit here" means that since the height increases a lot between t=4 and t=5, then the distance traveled that second is a lot and therefore the two people were walking fast while her statement "and then by less as they go along" means the height does not change much each second between t=5 and t=8 and therefore the distance traveled each of those seconds is successively less so that the two people are walking more slowly. Hence, they are slowing down. Either way, it seems clear that KG had come to understand that speed can be measured here in terms of a relationship between change in distance and change in time.

Discussion

These four precalculus students initially drew the same incorrect graph for Task 1. The graph they drew was the correct graph for Task 2, a graph showing the two people walking at the same steady pace. How did they come to see that their graph for Task 1 was incorrect and draw a correct graph? More relevantly, what did they come to understand about varying speed that helped them discover and correct their error? I will briefly highlight the

sequence of events for each of the four students and then form a general statement about coming to understand variable rate of change in this situation.

MH first noticed that if the two people were not walking at the same pace, then the graph could not be straight lines. Hence the graph had to be curved. She then came to see that curved graph meant, in the case of slowing down, more time to cover a particular distance. She then associated this with slope, steeper slope meaning faster pace and less steep slope meaning slower pace.

SCL realized that since his graph was not a curve, it showed the same speed. He knew his straight line graph showed steady pace since it took the same time to go a given distance. After drawing a correct graph for Task 1 and playing extensively with distances covered each second in the two graphs, he came to see that varying speed means covering a different number of feet each second, that slowing down means covering fewer feet each second and speeding up means covering more feet per second.

LC initially described her straight line segment graphs as showing slowing down and speeding up by the height of the graph decreasing and increasing, respectively. However, she came to see that height of the graph does not give information about speed but that change in height each second does. She then described slowing down and speeding up as less and more distance traveled, respectively, per second. Her resulting table of values yielded a graph that was not a straight line but was curved. It was here that she realized that varying speed means a curved, rather than a straight line, graph which she then described as not having a constant rate of change.

KG, like LC, believed initially that slowing down and speeding up means respectively that the graph decreases and increases. After much help she saw that straight line graphs show the same number of feet changing each second. So therefore varying speed should yield a curved graph. Careful examination of her curved graph showed that slowing down means

more seconds to cover fewer feet or more time to cover a given distance and speeing up means less time to go a certain distance or more feet covered each second.

These four students came to see that their initial straight line graphs were associated with constant speed since the number of feet changing each second is the same. This led to thinking about varying speed both in terms of a curved graph and a different number of feet changing each second with slowing down meaning fewer feet covered each second and speeding up meaning more feet covered each second. Some students moved from drawing a curved graph to making statements about slowing down and speeding up in terms of number of feet covered each second. Other students first formed relationships between speed and number of feet covered each second and then associated varying speed with a curved graph.

Two general conclusions can be made. One is that students approach varying speed in several ways. When they realized that straight lines do not mean varying speed, these four students pursued different routes to come to understand what varying speed meant. Some knew the graph must be a curve and then explored properties of a curve to see what could show slowing down and speeding up. Others characterized varying speed in terms of a different number of feet covered each second and then formed a curved graph that showed this. There may, of course, be other routes of coming to understand relationships between shape of graph and speed.

The second general conclusion is that shape of graph and changes over intervals represented powerful ways for these four precalculus students to think about constant and varying speed. If straight line means constant speed, then varying speed must be represented by curved graphs. If the same change in distance each second means constant speed, then varying speed must mean that the distance covered each second is not the same. These two connections, shape of graph and changes over intervals, seemed to be windows through which these four students viewed constant and then varying speed.

An instructional implication of this study is that teachers of calculus and precalculus should make more use of these ways of thinking about constant and varying speed. They should provide more opportunities for students to use their knowledge of shape of graph and changes over intervals to explore rate of change. More generally, teachers should notice knowledge students use to make sense of situations involving rate of change and design instruction that helps students use that knowledge to construct new knowledge of rate of change.

References

Ayers, T., Davis, G., Dubinsky, E., & Lewin, P. (1988). Computer experiences in learning composition of functions, Journal for Research in Mathematics Education, 19(3), 246-259.

Clement, J. (1989). The conception of variation and misconceptions in Cartesian graphing. Focus on Learning Problems in Mathematics, 11(1-2), 77-87.

Confrey (1980). Conceptual change, number concepts and the introduction to calculus. Unpublished doctoral dissertation, Cornell University, Ithaca, NY.

Cornu, B. (1981). Apprentissage de la notion de limite: Modeles spontanes et modeles propres. [Learning the limit concept: Individual and suitable models.] Actes du Cinquieme Colloque du Groupe Internationale PME (pp. 322-326). Grenoble: Universite de Grenoble.

Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. The Journal of Mathematical Behavior, 5, 281-303.

Dreyfus, T., & Eisenberg, T. (1982). Intuitive functional concepts : A baseline study on intuitions. Journal for Research in Mathematics Education, 13, 360-380.

Dreyfus, T. & Eisenberg, T. (1983). The function concept in college students: Linearity, smoothness, and periodicity. Focus on Learning Problems in Mathematics, 5,(3&4), 119-132.

Dreyfus, T. & Eisenberg, T. (1987). On the deep structure of functions. In J. C. Bergeron and C. Kieren (Eds.), Proceedings of the Eleventh International Conference of PME, Vol. I, Montreal, Canada, pp. 190-196.

Evan, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. Journal for Research in Mathematics Education, 24, 94-116.

Graham, K. G., & Ferrini-Mundy, J. (1989). An exploration of student understanding of central concepts in calculus. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco.

Hauger, G. (1991). College calculus students' understanding of functions. Unpublished manuscript.

Hauger, G. (1995). Rate of change knowledge in high school and college students: Growth of a biomass. Paper presented at the Annual Meeting of the American Educational Research Association, San Francsico.

Hauger, G. (1996.) Rate of change knowledge in high school and college students: Motion of a ball thrown up into the air. Paper presented at the Annual Meeting of the American Educational Research Association, New York.

Karplus, R. (1979). Continuous functions: Students' viewpoints. European Journal of Science Teaching, 1, 397-415.

McDermott, L., Rosenquist, M., & vanZee, E. (1987). Student difficulties in connecting graphs and physics: Examples from kinematics. American Journal of Physics, 55(6), 503-513.

Monaghan, J. (1991). Problems with the language of limits. For the Learning of Mathematics, 11(3), 20-24.

Monk, G. S., & Nemirovsky, R. (1992). The case of Dan: Student construction of a functional situation through visual attributes. Unpublished manuscript. University of Washington, Seattle.

Monk, G. S. (1988). Students' understanding of functions in calculus courses, In Humanistic Mathematics Network Newsletter (No. 2).

Monk, G. S. (1992a). Students' understanding of a function given a physical model. In Dubinsky, E. and Harel, G. (Eds.), The concept of function: Aspects of epistemology and pedagogy: MAA notes, vol. 25. Washington, DC: MAA.

Monk, G. S. (1992b). A study of calculus students' constructions of functional situations: The case of the shadow problem. Paper presented at the annual meeting of the American Educational Research Association, San Francisco.

Nemirovsky,R. (1991). Notes about the relationship between the history and the constructive learning of calculus. Proceedings of the Segundo Simposia Internacional sobre Investigacion en Educacion Matematica, (pp. 37-54). Universidad Autonoma del Estado va Mexico, Cuernacava, Mexico.

Nemirovsky, R. (1994). On ways of symbolizing: The case of Laura and the velocity sign. Journal of Mathematical Behavior, 13(4), 389-422.

Nemirovsky, R., & Rubin, A. (1991). "It makes sense if you think about how the graph works. But in reality ...". In F. Furinghetti (Ed.), Proceedings of the Fifteenth Annual Meeting, North American Chapter of the International Group for the Psychology of Mathematics Education, Vol. 3 (pp. 57-64).

Nemirovsky, R., & Rubin, A. (1992). Students' tendency to assume resemblances between a function and its derivative. Cambridge, MA: TERC.

Orton, A. (1983a). Students' understanding of differentiation. Educational Studies in Mathematics, 15, 235-250.

Orton, A. (1983b). Students' understanding of integration. Educational Studies in Mathematics, 14, 1-18.

Orton, A. (1984). Understanding rate of change. Mathematics in School, 13(5), 23-26.

Rubin, A., & Nemirovsky, R. (1991). Cars, computers, and air pumps: Thoughts on the roles of physical and computer models in learning the central concepts of calculus. In R. G. Underhill (Ed.), Proceedings of the Thirteenth Meeting of the North American Conference for the Psychology of Mathematics Education (PME-NA), Vol. 2 (pp. 168-174). Blacksburg, VA: Division of Curriculum & Instruction.

Smith, J. P. (1990). Learning rational number. Unpublished doctoral dissertation, University of California, Berkeley.

Tall, D. O. (1987). Constructing the concept image of a tangent. In J. Bergeron, N Herscovics, & C. Kieren (Eds.), Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education, Vol. 3 (pp. 69-75.) Montreal, Canada: Universite de Montreal.

Tall, D. O., & Schwarzenberger, R. L. (1978). Conflicts in the learning or real numbers and limits. Mathematics Teaching, 83, 44-49.

Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with particular reference to limits and continuity. Educational Studies in Mathematics, 12, 151-169.

Thompson, P. W., & Thompson, A. G. (1992). Images of rate. Paper presented at the Annual Meeting of the American Educational Research Association, San Francisco.

Thompson, P. W. (1994a). Images of rate and operational understanding of the fundamental theorem of calculus. Educational Studies in Mathematics, 26, 229-274.

Thompson, P. W. (1994b). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Ed.s), The development of multiplicative reasoning, (pp. 179-234). New York: SUNY Press.

Vinner, S. (1982). Conflicts between definitions and intuitions: The case of the tangent. In A. Vermandel (Ed.), Proceedings of the Sixth International Conference for the Psychology of Mathematics Education (pp. 24-28). Antwerp, Belgium: Universitaire Instelling Antwerpen.

Vinner, S. (1983). Concept definition, concept image and the notion of function. International Journal of Mathematical Education in Science and Technology, 14, 239-305.

Vinner, S. (1987). Continuous functions - images and reasoning in college students. In J. Bergeron (Ed.) Proceedings of the Eleventh International Conference on the Psychology of Mathematics Education, Montreal.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of a function. Journal for Research in Mathematics Education, 20, 356-366.

Williams, S. (1991). Models of limit held by college calculus students. Journal for Research in Mathematics Education, 22(3), 219-236.

White, P., & Mitchelmore, M. (1996). Conceptual knowledge in introductory calculus. Journal for Research in Mathematics Education, 27(1), 79-95.

Appendix A - Statements of tasks

Task 1 - Two people start at opposite corners of a room and walk toward each other. As they walk, they both slow down as they get closer to each other, pass, and then they both speed up as they get farther apart. This takes a total of eight seconds. The opposite corners of the room are 20 feet apart.

a. On the graph paper supplied, draw a graph showing the distance between the two people at each moment in time. Describe your graph.

b. How does your graph show the two people slowing down? Speeding up?

c. Now complete the table of values for your graph.

d. How does your table of values show the two people slowing down? Speeding up?

Task 2 - These same two people decide again to start at opposite corners of the room and walk toward each other but this time they both decide to maintain the same steady pace the whole way. Again it takes a total of eight seconds for each to walk the 20 feet.

a. On the same graph paper you used for Task 1, draw a graph showing the distance between the two people at each moment in time. Describe your graph.

b. How does your graph show the steady pace of the two people?

c. Now complete the table of values for your graph.

d. How does your table of values show the steady pace of the two people?

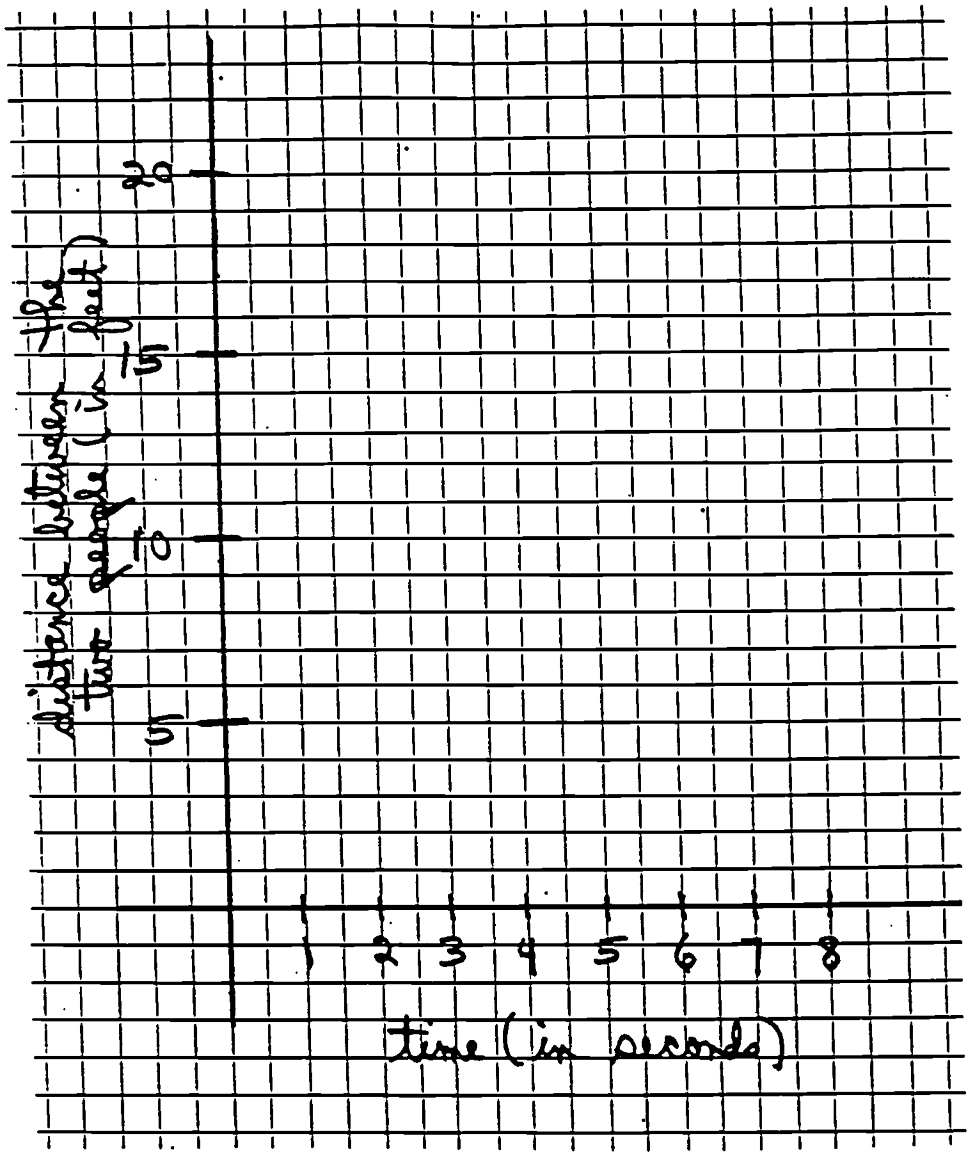
Task 3 - These same two people decide once more to start at opposite corners and walk toward each other. But this time as they walk, they both speed up as they get closer to each other, pass, and then they both slow down as they get farther apart. Once again it takes eight seconds for each to walk the 20 feet.

a. On the same graph paper you used for Tasks 1 and 2, draw a graph showing the distance between the two people at each moment in time. Describe your graph.

b. How does your graph show the two people speeding up? Slowing down?

c. Now complete the table of values for your graph.

d. How does your table of values show the two people speeding up? Slowing down?



Task 1

Time	Distance
0 seconds	
1 second	
2 seconds	
3 seconds	
4 seconds	
5 seconds	
6 seconds	
7 seconds	
8 seconds	

Task 2

Time	Distance
0 seconds	
1 second	
2 seconds	
3 seconds	
4 seconds	
5 seconds	
6 seconds	
7 seconds	
8 seconds	

Task 3

Time	Distance
0 seconds	
1 second	
2 seconds	
3 seconds	
4 seconds	
5 seconds	
6 seconds	
7 seconds	
8 seconds	

Appendix B - Seven basic shapes of graphs with tables

Graph 1

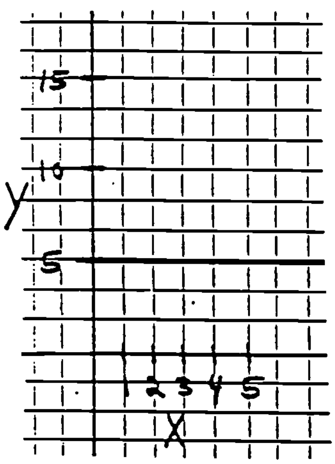


Table 1

x	y
0	5
1	5
2	5
3	5
4	5
5	5

Graph 2

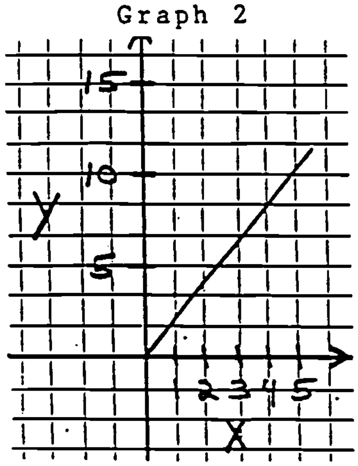


Table 2

x	y
0	0
1	2
2	4
3	6
4	8
5	10

Graph 3

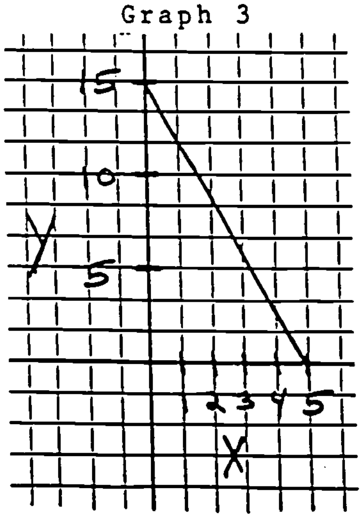


Table 3

x	y
0	15
1	12
2	9
3	6
4	3
5	0

Graph 4

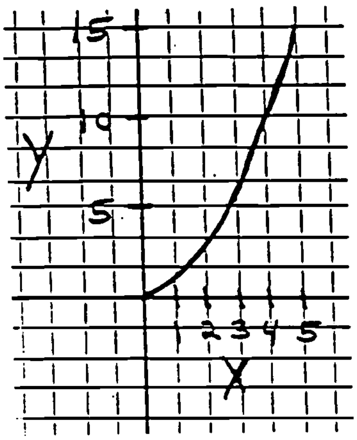


Table 4

x	y
0	0
1	1
2	3
3	6
4	10
5	15

Graph 5

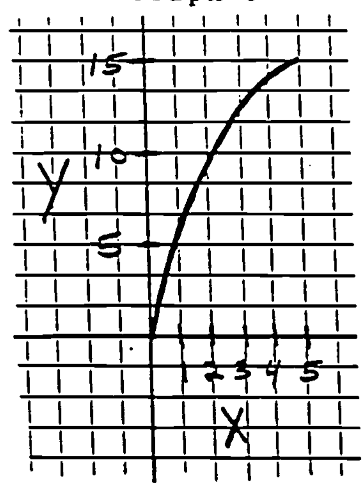


Table 5

x	y
0	0
1	5
2	9
3	12
4	14
5	15

Graph 6

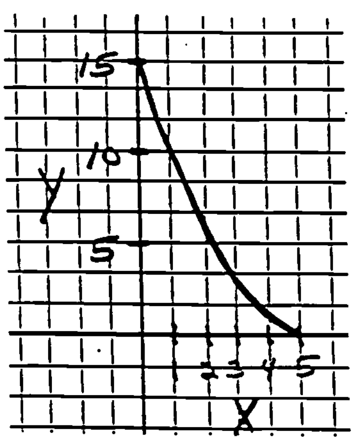


Table 6

x	y
0	15
1	10
2	6
3	3
4	1
5	0

Graph 7

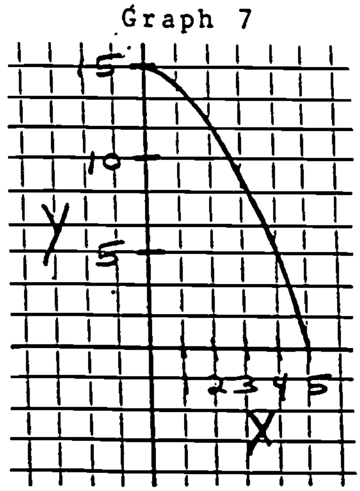
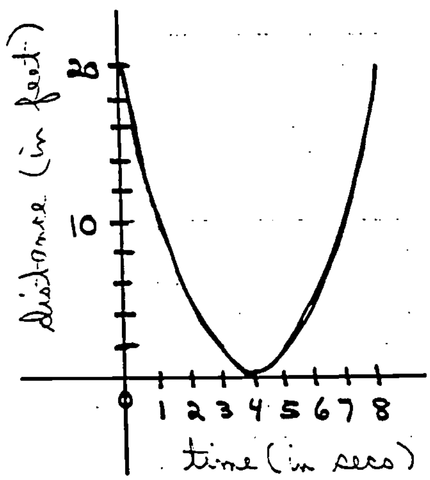


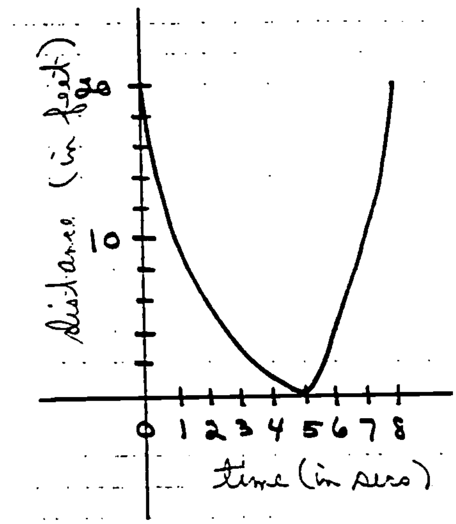
Table 7

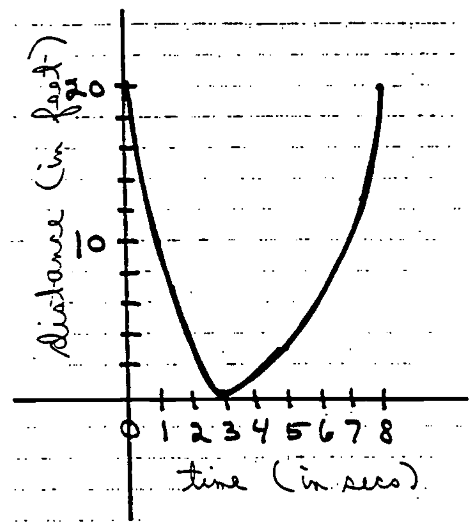
x	y
0	15
1	14
2	12
3	9
4	5
5	0

Appendix C - Correct graphs and tables for Tasks 1, 2, and 3

Task 1







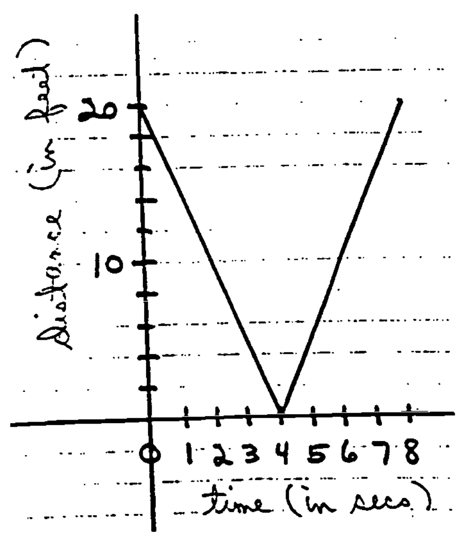
Time	Distance
0 seconds	20 feet
1 second	10 feet
2 seconds	5 feet
3 seconds	2 feet
4 seconds	0 feet
5 seconds	2 feet
6 seconds	5 feet
7 seconds	10 feet
8 seconds	20 feet

Time	Distance
0 seconds	20 feet
1 second	10 feet
2 seconds	6 feet
3 seconds	3 feet
4 seconds	1 foot
5 seconds	0 feet
6 seconds	4 feet
7 seconds	10 feet
8 seconds	20 feet

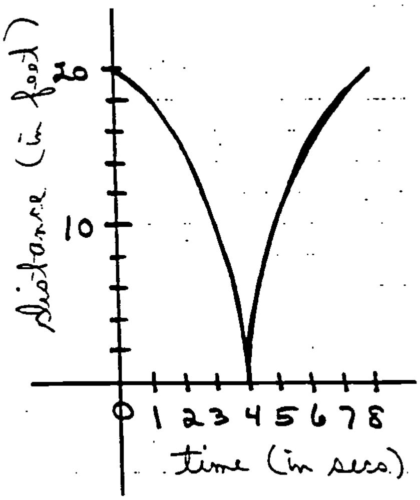
Time	Distance
0 seconds	20 feet
1 second	10 feet
2 seconds	4 feet
3 seconds	0 feet
4 seconds	1 foot
5 seconds	3 feet
6 seconds	6 feet
7 seconds	10 feet
8 seconds	20 feet

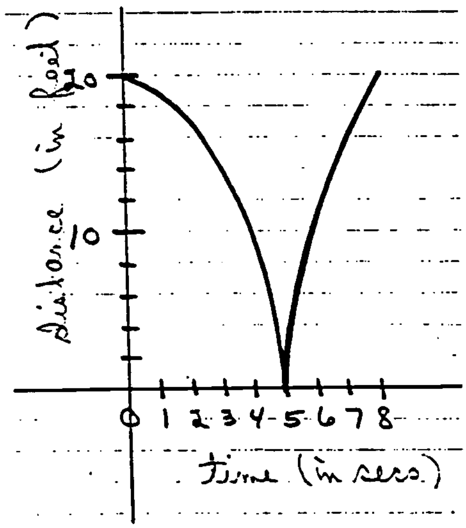
Task 2

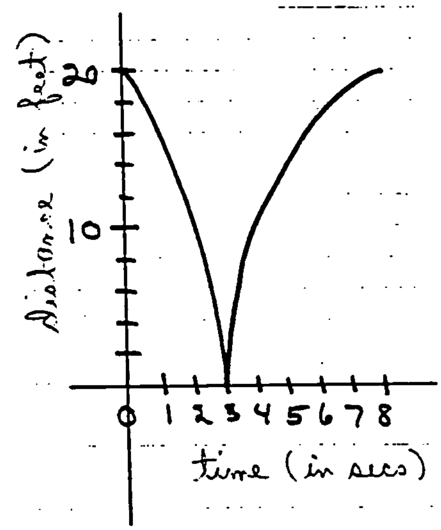
Time	Distance
0 seconds	20 feet
1 second	15 feet
2 seconds	10 feet
3 seconds	5 feet
4 seconds	0 feet
5 seconds	5 feet
6 seconds	10 feet
7 seconds	15 feet
8 seconds	20 feet



Task 3







Time	Distance
0 seconds	20 feet
1 second	18 feet
2 seconds	15 feet
3 seconds	10 feet
4 seconds	0 feet
5 seconds	10 feet
6 seconds	15 feet
7 seconds	18 feet
8 seconds	20 feet

Time	Distance
0 seconds	20 feet
1 second	19 feet
2 seconds	17 feet
3 seconds	14 feet
4 seconds	10 feet
5 seconds	0 feet
6 seconds	10 feet
7 seconds	16 feet
8 seconds	20 feet

Time	Distance
0 seconds	20 feet
1 second	16 feet
2 seconds	10 feet
3 seconds	0 feet
4 seconds	10 feet
5 seconds	14 feet
6 seconds	17 feet
7 seconds	19 feet
8 seconds	20 feet

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February 21, 1997

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